Probability and Optimization Models for Racing

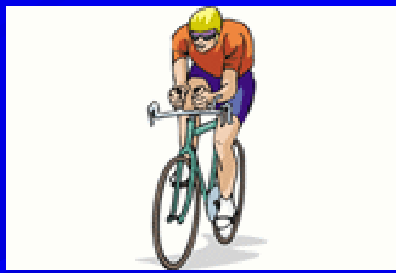
Victor S. Y. Lo
University of British Columbia
Fidelity Investments

Disclaimer: This presentation does not reflect the opinions of Fidelity Investments. The work here was completed at University of British Columbia and the University of Hong Kong.

Outline

- Areas to Discuss in Racing:

- Favorite-longshot Bias (Economics, Statistics)



- Ordering Probabilities and Optimal Investment (Probability, Statistics, Finance)

Part 1: Favorite-Longshot Bias

- Favorites are underbet and longshots are overbet – Busche & Hall (1988), Ali (1977)
- A well-known phenomenon in economic literature - Ziemba (2004); Creating opportunities – Bolton & Chapman (1986), Bentor (1994)
- Economic interpretation: Risk-loving behavior

Favorite-Longshot Bias

Define:

P_i = Bet fraction on horse i, i.e. consensus win probability, i = 1, …, n
= (1- track take)/(1 + O_i), where O_i = Odds on i

π_i = *objective (true) win probability of i*

$$\pi_i < P_i \quad if \quad \pi_i's \; are \quad low$$

$$\pi_i > P_i \quad if \quad \pi_i's \; are \quad high$$

Statistical Model

- Many techniques mentioned in the literature – Ali (1977), Asch and Quandt (1984)
- Propose to use a simple logit model, Bacon-Shone, Lo, and Busche (1992a):

$$\pi_i = \frac{P_i^{\beta}}{\sum_{j=1}^{n} P_j^{\beta}}$$

Statistical Model (continued)

- $\beta > 1 \rightarrow$ risk-prefer
- $\beta = 1 \rightarrow$ risk-neutral
- $\beta < 1 \rightarrow$ risk-averse

$$\pi_i = \frac{P_i^{\beta}}{\sum_{j=1}^{n} P_j^{\beta}}$$

Universal Comparisons

US racetracks consistently have a risk-prefer bias with $\beta > 1$

Racetrack	# races	Estimated β	p-value for H1: β *n.e.* 1	Average pool size
US (Quandt's 83-84):				
Atlantic City	712	1.10	0.08	unknown
Meadowlands	705	1.12	0.02	$52K
US (Ali's 70-74):				
Saratoga	9,072	1.16	~0	$25K
Roosevlt	5,806	1.13	~0	$218K
Yonkers	5,369	1.13	~0	$228K
Japan (90)	1,607	1.07	0.01	$168K
Hong Kong (81-89):				
Happy Valley	2,212	1.04	0.25	$1.1M
Shatin	1,943	0.94	0.04	$1.1M
China (23-35):				
Shanghai	730	1.03	0.38	unknown

Utility Function Interpretation

- Expected utility maximizer is indifferent between betting on any horses in a race:

See Ali (1977), Lo (1992)

- It can be shown:

$$E(U_i) = K \quad \forall i = 1, \ldots, n.$$

$$U(1+O_i) = K\left[1 + \frac{\sum_{j \neq i} P_j^{\beta}}{(1+t)^{\beta}}(1+O_i)^{\beta}\right] \propto \underbrace{(1+O_i)^{\beta}}$$

Power utility

Then, Arrow–Pratt Measure of Absolute Risk Aversion

$$= {-U''(x)}\big/{U'(x)} = -(\beta - 1)/x \qquad (2)$$

< 0, and increases with wealth, if $\beta > 1$

$\Rightarrow$ Bettors take more risk as capital decline "Risk-lovers"

Conclusion and Research Opportunities

- Favorite-longshot bias exists in many US racetracks (but not huge)…
- …but does not exist in some Asian racetracks – would it depend on the *Pool Size*?
- Bias in other investment areas – see Ziemba (2004)
- Opportunity to understand bias or accuracy in complicated bets, e.g. Lo and Busche (1994)
- Opportunity to apply similar logit models in other applications and other sports, e.g. Lo (1994a), Willoughby (2002)

Part 2: Ordering Probabilities

- Running time distribution (T_i's) is key to determine ordering probabilities:

$$\pi_i = P(T_i < \underset{r \neq i}{MIN} \{T_r\})$$

$$= \int_0^{\infty} \prod_{r \neq i} [1 - F(t_i \mid \theta_r)] f(t_i \mid \theta_i) dt_i, \qquad (3)$$

$where \quad \theta_i = E(T_i) \; or \; location \quad parameter \quad ,$

$and \quad f(.) \; and \quad F(.) \; are \quad pdf \quad and \quad cdf \quad , resp \; .$

$Given \quad \pi_i ' s, solve \quad for \; \theta_i ' s. Then \quad compute \quad :$

$$\pi_{ij} = P(T_i < T_j < \underset{r \neq i,j}{MIN} \{T_r\})$$

$$= \int_0^{\infty} F(t_j \mid \theta_i) \prod_{r \neq i,j} [1 - F(t_j \mid \theta_r)] f(t_j \mid \theta_j) dt_j \qquad (4)$$

Running Time Distribution

- The following types have been considered in literature, all assuming *independent* running times:

$$Exponential - Harville(1973):\ f(t_i \mid \theta_i) = \frac{1}{\theta_i}\exp(-t_i/\theta_i)$$

$$Normal - Henery(1981): \quad T_i \sim N(\theta_i, 1)$$

$$Gamma - Stern(1990): \quad T_i \sim Gamma(r, \theta_i),$$

$$where\ r\ is\ the\ shape\ parameter$$

Exponential Running Time

- Strictly speaking, we only need g(T) ~ Exponential, where g(.) is a monotonically increasing function

$$\pi_{ij} = P(i \text{ finishes } 1st \text{ and } j \text{ finishes } 2nd)$$

$$= \frac{\pi_i \pi_j}{1-\pi_i}, \qquad (5)$$

$$\text{where } \pi_i \text{ can be estimated by bet fraction } P_i$$

$$\pi_{ijk} = P(i \text{ finishes } 1st, j \text{ finishes } 2nd, k \text{ finishes } 3rd)$$

$$= \frac{\pi_i \pi_j \pi_k}{(1-\pi_i)(1-\pi_i-\pi_j)} \qquad (6)$$

Normal and Gamma Running Time

- The formulas are complex, as one has to solve (3), a system of integral equations, for θ_i's, and then compute (4)

- Henery(1981) proposed to use a first-order Taylor series approximation under normal running time

- Lo and Bacon-Shone (2007) proposed a simple approximation…

Simple Approximation

Lo and Bacon-Shone (2007):

$$\pi_{ijk} = \pi_i \frac{\pi_j^{\lambda}}{\sum_{s \neq i} \pi_s^{\lambda}} \frac{\pi_k^{\tau}}{\sum_{t \neq i,j} \pi_t^{\tau}} \qquad (7)$$

$where\ \pi_i's\ can\ be\ estimated\ by\ bet\ fractions\ P_i's,$

$\lambda\ and\ \tau\ are\ parameter\ values\ in\ Lo\ and\ Bacon-Shone(2007).$

$Note\ that\ for\ Exponential\ time,\ \lambda = \tau = 1,$

$(7)\ reduces\ to\ (5)\ and\ (6).$

Running Time Distribution Competiton

- So, which distribution should be used?

- Lo and Bacon-Shone (1994) found that Harville model has a systematic bias in estimating ordering probabilities based on Hong Kong data and Henery model is clearly superior
- Bacon-Shone, Lo, and Busche (1992b) had a similar conclusion using Meadowlands data, however…
- … Lo (1994b) found that Stern model with r=4 is better than both Henery and Harville using Japan data!

Correlated Running Times

Constant correlation, i.e. $\text{Corr}(T_i, T_j) = \rho \; \forall i \neq j$,

reduces to Henery; more complicated cases:

A) Non-constant correlation: $\rho_{ij} = \psi_i \psi_j \;\; \forall i \neq j$,

where $\log(\frac{\psi_i}{1-\psi_i}) = -\delta - \gamma(\theta_i - \bar{\theta}),\; \bar{\theta} = \frac{1}{n}\sum_i \theta_i$,

i.e. correlations higher for stronger pairs.

B) Non-constant variance: $\sigma_i = \exp[\kappa(\theta_i - \bar{\theta})]$,

i.e., if $\kappa > 0$, weaker horses will have higher variance.

If $\gamma = \kappa = 0$, it reduces to Henery.

Empirical Results

- First order Taylor series approx employed for

Model	Estimates	p-value of Lik ratio test rel to Henery
A) Non-constant correlation (γ only)	$\gamma = 0.58$	0.06
A) Non-constant correlation (γ and δ)	$\gamma = 0.60$, $\delta = 0.05$	0.18
B) Non-constant variance	$\kappa = 0.08$	0.06

Non-constant correlation with slope γ only or non-constant variance shows some promise

Kelly Criterion for Optimal Investment

- Instead of mean-variance criterion, we maximize expected log wealth → growth rate

$$\text{Wages on all opportunities in race t}, (X^*_{t1},\ldots,X^*_{tm})^T$$

$$= \arg\max_{X_{t1},\ldots X_{tm}} \{E(\log W_t) \mid \sum_i X_{ti} \le W_{t-1}, X_{ti} \ge 0\, \forall i\}$$

$$\text{where } W_t = \text{total wealth at the end of race t.}$$

- Breiman(1960), Thorp(1971), Algoet & Cover(1988) show long-run asymptotic optimality
- Adopted by Hausch, Ziemba, & Rubinstein(1981) using exponential running times, and Lo, Bacon-Shone, & Busche(1995) and Hausch, Lo, & Ziemba (1994) using other running time distributions, all showed promises

Conclusion and Research Opportunities

- Knowing the appropriate running time distribution is key to determining ordering probabilities
- There appears to be no universal best distribution but Henery (Normal) and Stern (Gamma) are competitive
- Simple approximation is available for Henery and Stern
- Correlated running time model is more complex but may be better
- Other approximation methods may be considered especially for more complicated models
- (Fractional) Kelly is promising for optimal betting

References for Favorite-Longshot Bias

- Ali, M.M. (1977) "Probability and Utility Estimates for Racetrack Bettors," *J. of Political Economy*, 84, p.803-815.
- Asch, P., Malkiel, B., and Quandt, R. (1984) "Market Efficiency in Racetrack Betting," *J. of Business* 57, p.165-174.
- Bacon-Shone, J., Lo, V.S.Y., and Busche, K. (1992a) "Modelling the Winning Probability," *Research Report* 10, Dept. of Statistics, the University of Hong Kong.
- Benter, W. (1994) "Computer Based Horse Race Handicapping and Wagering Systems: A Report," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.183-198.
- Bolton, R.N. and Chapman, R.G. (1986) "Searching for Positive Returns at the Track, A Multinomial Logit Model for Handicapping Horse Races," *Management Science*, 32, p.1040-1059. Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.237-247.
- Busche, K. and Hall, C.D. (1988) "An Exception to the Risk Preference Anomaly," *J. of Business,* 61, p.337-346.
- Lo, V.S.Y. (1992) "Statistical Modelling of Gambling Probabilities," *PhD Thesis*, Dept. of Statistics, The University of Hong Kong
- Lo, V.S.Y. (1994a) "Application of Logit Models in Racetrack Data," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.307-314.
- Lo, V.S.Y. and Busche, K. (1994) "How Accurately do Betters Bet in Doubles?," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.465-468.
- Willoughby, K.A. (2002) "Winning Games in Canadian Football: A Logistic Regression Analysis," *The College Mathematics Journal*, 33, No.3, p.215-220.
- Ziemba, W.T. (2004) "Behavioral Finance, Racetrack Betting and Options and Futures Trading," *Mathematical Finance Seminar*, Stanford University.

References for Ordering Probabilities

- Bacon-Shone, J.H., Lo, V.S.Y., and Busche, K. (1992b) "Logistic Analyses of Complicated Bets," *Research Report 11*, Dept. of Statistics, the University of Hong Kong.
- Harville, D.A. (1973) "Assigning Probabilities to the Outcomes of Multi-Entry Competitions," *J. of American Statistical Association*, 68, p.312-316.
- Hausch, DB., Ziemba, W.T., and Rubinstein, M. (1981) "Efficiency of the Market for Racetrack Betting," *Management Science,* 27, p.1435-1452.
- Henery, R.J. (1981) "Permutation Probabilities as Models for Horse Races," *J. of Royal Statistical Society B*, 43, p.86-91.
- Henery, R.J. (1985) "On the Average Probability of Losing Bets on Horses with Given Starting Price Odds," *J. of Royal Statistical Society A,* 148, p.342-349.
- Lo, V.S.Y. (1994b) "Application of Running Time Distribution Models in Japan," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.237-247.
- Lo, V.S.Y. and Bacon-Shone, J. (1994) "A Comparison between Two Models for Predicting Ordering Probabilities in Multiple-Entry Competitions," *The Statistician*, 43, No.2, p.317-327.
- Lo, V.S.Y. and Bacon-Shone, J. (2007) "Approximating the Ordering Probabilities of Multi-Entry Competitions By a Simple Method," To appear in: Hausch, D.B. and Ziemba, W.T. ed. (2007) *Handbook of Investments: Efficiency of Sports and Lottery Markets,* Elsevier.
- Stern, H. (1990) "Models for Distributions on Permutations," *J. of American Statistical Association,* 85, p.558-564.

References for Optimal Investment Strategy

- Algoet, P.H. and Cover T.H. (1988) "Asymptotic Optimality and Asymptotic Equipartition Properties of Log-optimum Investment," *The Annals of Probability,* 16, No.2, p.876-898.
- Benter, W. (1994) "Computer Based Horse Race Handicapping and Wagering Systems: A Report," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.183-198.
- Breiman, L. (1960) "Investment Policies for Expanding Businesses Optimal in a Long-run Sense," *Naval Research Logistics Quarterly*, *7*, p.647-651.
- Haigh, J. (2000) "The Kelly Criterion and Bet Comparisons in Spread Betting," *The Statistician*, 40, Part 4, p.531-539.
- Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. (1994) "Pricing Exotic Racetrack Wagers," in Hausch, D.B., Lo, V.S.Y., and Ziemba, W.T. ed. (1994) *Efficiency of Racetrack Betting Markets,* Academic Press, p.469-483.
- Hausch, DB., Ziemba, W.T., and Rubinstein, M. (1981) "Efficiency of the Market for Racetrack Betting," *Management Science,* 27, p.1435-1452.
- Lo, V.S.Y., Bacon-Shone, J., and Busche, K. (1995) "The Application of Ranking Probability Models to Racetrack Betting," *Management Science,* 41, p.1048-1059.
- Thorp E.O. (1971) "Portfolio Choice and the Kelly Criterion," *Business and Economics Statistics Section, Proceedings of the American Statistical Association.*
- Ziemba, W.T. (2004) "Behavioral Finance, Racetrack Betting and Options and Futures Trading," *Mathematical Finance Seminar*, Stanford University.